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AUTHOR(S):

Akiyama, Shigeki

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# A remark on almost uniform distribution modulo 1

Shigeki Akiyama ( 秋山茂樹, 新潟大理 )

Let  $(a_n)$ ,  $n = 1, 2, \dots$  be a sequence of real numbers and  $A(I, (a_n), N)$  be the *counting function*, that is, the number of  $n = 1, 2, \dots, N$  that  $\{a_n\}$  is contained in a certain interval  $I \subset [0, 1]$ . Here we denote by  $\{a_n\} = a_n - [a_n]$ , the fractional part of  $a_n$ . First we recall a kind of generalization of the classical definition of uniform distribution modulo 1 (see [11], [3] and [10]).

**Definition.** The sequence  $(a_n)$  is said to be *almost uniformly distributed modulo 1* (abbreviated a.u.d. mod 1) if there exist a strictly increasing sequence of natural numbers  $(n_j)$ ,  $j = 1, 2, \dots$  and, for every pair of  $a, b$  with  $0 \leq a < b \leq 1$ ,

$$\lim_{j \rightarrow \infty} \frac{A([a, b), (a_n), n_j)}{n_j} = b - a.$$

The purpose of this note is to emphasize the usefulness of this concept in considering the oscillation problems in number theory.

Now, for example, we define  $(c_n)$  by

$$c_n = \frac{n}{2^{1 + [\log_2 n]}},$$

Then  $(c_n)$  is a.u.d. mod 1 but not u.d. mod 1. It is obvious that if the sequence  $(a_n)$  is u.d. mod 1, then a.u.d. mod 1. On the contrary, if

$$n_{j+1} - n_j = o(n),$$

then a.u.d. mod 1 implies u.d. mod 1. According to the classical method of uniform distribution theory (see e.g. [8]), we can show the following

**Proposition 1.** The sequence  $(a_n)$ ,  $n = 1, 2, \dots$  is a.u.d. mod 1 if and only if there exist a strictly increasing sequence of natural numbers  $(n_j)$ ,  $j = 1, 2, \dots$ , and for every real-valued continuous function on the interval  $[0, 1]$ , we have

$$\lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{i=1}^{n_j} f(\{a_i\}) = \int_0^1 f(x) dx.$$

**Proposition 2. (Weyl's Criterion for a.u.d. mod 1)** The sequence  $(a_n)$ ,  $n = 1, 2, \dots$  is a.u.d. mod 1 if and only if there exist a strictly increasing sequence of natural numbers  $(n_j)$ ,  $j = 1, 2, \dots$ , and for every integer  $h$ , we have

$$\lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{i=1}^{n_j} \exp(2\pi h \sqrt{-1} \{a_i\}) = 0.$$

We should pay attention to the next generalization of Fejér's Theorem.

**Theorem 1. (Fejér's Theorem for a.u.d. mod 1)** Let  $(f(n))$ ,  $n = 1, 2, \dots$  be a sequence of real numbers and  $\Delta f(n) = f(n+1) - f(n)$ . If the following three conditions is satisfied, then  $(f(n))$  is a.u.d. mod 1:

1. There exists a natural number  $N$  that  $\Delta f(n)$  is monotone when  $n \geq N$  (hereafter, we say this property as *ultimately monotone*),
2.  $\lim_{n \rightarrow \infty} \Delta f(n) = 0$ ,
3.  $\limsup_{n \rightarrow \infty} n|\Delta f(n)| = \infty$ .

Note that the corresponding third conditions for u.d. mod 1 is:

$$\lim_{n \rightarrow \infty} n|\Delta f(n)| = \infty.$$

Moreover, it is shown in [7] that  $\limsup_{n \rightarrow \infty} n|\Delta f(n)| = \infty$  is the necessity condition for u.d. mod 1 (see also [6]). Concerning this fact, in [3], it is shown that  $(\log n)$  is not a.u.d. mod 1 but a.u.d. mod 1 in the "average" sense. It is an interesting problem to study this delicate difference between u.d. mod 1 and a.u.d. mod 1. We can show the following:

**Corollary 1.** Let  $(g(n))$  be a sequence of real numbers which satisfies three conditions:

- (C1)  $g(n) = o(n)$ ,
- (C2) Let  $f(n) = \frac{1}{n} \sum_{k=1}^n g(k)$ , then  $f(n)$  is *not* a.u.d. mod 1,
- (C3)  $\limsup_{n \rightarrow \infty} |f(n) - g(n+1)| = \infty$ .

Then  $\Delta^2 f(n)$  changes its sign infinitely many times. Here  $\Delta^2 f(n) = \Delta(\Delta f(n))$ .

*Proof.* We have

$$\begin{aligned}\Delta f(n) &= \frac{1}{n+1} \sum_{k=1}^{n+1} g(k) - \frac{1}{n} \sum_{k=1}^n g(k) \\ &= \frac{1}{n+1} g(n+1) - \frac{1}{n(n+1)} \sum_{k=1}^n g(k).\end{aligned}\tag{1}$$

This shows that  $\lim_{n \rightarrow \infty} \Delta f(n) = 0$ . And by (1),

$$(n+1)\Delta f(n) = g(n+1) - f(n).$$

Thus

$$\limsup_{n \rightarrow \infty} n|\Delta f(n)| = \infty.$$

If  $\Delta f(n)$  is ultimately monotone, then  $f(n)$  is a.u.d. mod 1, which contradicts with the assumption.  $\square$

Let  $P_n$  be the  $n$ -th prime and we will later apply this Corollary 1 for the oscillation problem of  $P_n$ . Now we show

**Theorem 2.**  $(\log P_n)$  is not a.u.d. mod 1.

*Proof.* Let  $\mathbf{T} = \mathbf{R}/\mathbf{Z}$  be the real torus, which is identified with the interval  $[0, 1)$  via the map  $x \rightarrow \{x\}$ . Define by  $\chi_\tau$ , the characteristic function of  $[\tau, \tau + 1/2) \bmod \mathbf{Z}$  in  $\mathbf{T}$ , and by  $\pi(x)$ , the number of primes smaller or equal to  $x$ . Let us evaluate, for a positive integer  $k$ ,

$$T_\tau(k) = \frac{1}{\pi(N_k)} \sum_{n=1}^{\pi(N_k)} \chi_\tau(\log P_n),$$

with  $N_k = e^{k+\tau+1/2}$ . By using the prime number theorem of the form:

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

we have, for a positive number  $c = o(k)$ ,

$$\begin{aligned}\pi(N_k)T_\tau(k) &= \sum_{n=1}^{\pi(N_{k-c})} + \sum_{n=\pi(N_{k-c+1})}^{\pi(N_k)} \\ &= \frac{e^{k+\tau+1/2}}{k+\tau+1/2} - \frac{e^{k+\tau}}{k+\tau} + \frac{e^{k+\tau-1/2}}{k+\tau-1/2} - \frac{e^{k+\tau-1}}{k+\tau-1} + \dots \\ &\quad + \frac{e^{k+\tau+3/2-c}}{k+\tau+3/2-c} - \frac{e^{k+\tau+1-c}}{k+\tau+1-c} + O\left(\frac{e^{k-c}}{k-c}\right) + O\left(\frac{c \cdot e^k}{(k-c)^2}\right).\end{aligned}$$

If  $c = [2 \log k]$ , then

$$\begin{aligned}
 T_\tau(k) &= 1 - \frac{k + \tau + 1/2}{e^{1/2}(k + \tau)} + \frac{k + \tau + 1/2}{e(k + \tau - 1/2)} - \frac{k + \tau + 1/2}{e^{3/2}(k + \tau - 1)} + \dots \\
 &\quad + \frac{k + \tau + 1/2}{e^{c-1}(k + \tau + 3/2 - c)} - \frac{k + \tau + 1/2}{e^{c-1/2}(k + \tau + 1 - c)} + O\left(\frac{\log k}{k}\right) \\
 &= 1 - e^{-1/2} + e^{-1} - e^{-3/2} + \dots + e^{-c+1} - e^{-c+1/2} + O\left(\frac{\log k}{k}\right) \\
 &= \frac{\sqrt{e}}{1 + \sqrt{e}} + O\left(\frac{\log k}{k}\right).
 \end{aligned}$$

Here, the implied constant of the last  $O$  symbol does not depend on the choice of  $\tau \in [0, 1]$ , and we have  $\sqrt{e}/(1 + \sqrt{e}) = 0.622459\dots > 1/2 + \delta$  with a positive constant  $\delta$ . This shows that there exist an open neighborhood of  $\tau$  in  $\mathbf{T}$ :

$$U_\tau = \{x \in \mathbf{T} \mid d(x, \tau) < \epsilon\}$$

that if  $\{\log N\} - 1/2 \in U_\tau$ , then

$$\frac{1}{N} \sum_{n=1}^N \chi_\tau(\log P_n) > 1/2 + \delta, \quad (2)$$

for sufficiently large  $N$ . Here  $d(x, y)$  is the natural distance between  $x$  and  $y$  on  $\mathbf{T}$ . Remark that the value  $\epsilon$  does not depend on the choice of  $k$ , if  $k$  is sufficiently large. As  $\mathbf{T}$  is compact, there exist a finite sub covering  $\mathbf{T} \subset \cup_{i=1}^m U_{\tau_i}$ . Thus there exist a constant  $M$  that if  $N \geq M$ , then we have (2). If the sequence  $(\log P_n)$  is a.u.d. mod 1, then by Proposition 1, there exist a strictly increasing sequence of natural numbers  $(n_j)$ ,  $j = 1, 2, \dots$  that

$$\lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{i=1}^{n_j} \chi_\tau(\{\log P_i\}) = \frac{1}{2},$$

which is a contradiction. □

Now we give a very different proof of the results of [2].

**Theorem 3.**  $\Delta^2 \log P_n$  changes its sign infinitely many times.

*Proof.* Let  $g(n) = n \log P_n - (n-1) \log P_{n-1}$  and  $f(n) = \log P_n$  in Corollary 1. (Here we put  $P_0 = 1$  for example.) By using Theorem 2, it suffice to show (C1) and (C3). By using prime number theorem, we have

$$\log P_n = 1 + \frac{P_n}{n} + O\left(\frac{1}{\log P_n}\right). \quad (3)$$

Thus we see

$$\begin{aligned} g(n) &= P_n - P_{n-1} + 1 + O\left(\frac{n}{\log n}\right) \\ &= o(n). \end{aligned}$$

Here we used the fact:

$$P_n - P_{n-1} = O(P_n^\theta),$$

with a certain positive constant  $\theta < 1$ . This type of result was first shown by G. Hoheisel in [4] with  $\theta = 1 - 1/33000 + \epsilon$ . The best knowledge up to now is  $\theta = 23/42$  in [5]. For the condition (C3),

$$\begin{aligned} g(n+1) - f(n) &= (n+1)(\log P_{n+1} - \log P_n) \\ &> \frac{(n+1)(P_{n+1} - P_n)}{P_{n+1}} \\ &\sim \frac{P_{n+1} - P_n}{\log P_n}. \end{aligned}$$

Here we write  $f \sim g$  if  $|f/g| \rightarrow 1$ . P. Erdős [1] was the first to obtain

$$\limsup_{n \rightarrow \infty} \frac{P_{n+1} - P_n}{\log P_n} = \infty,$$

by showing

$$\limsup_{n \rightarrow \infty} \frac{(P_{n+1} - P_n)(\log \log \log P_n)^2}{\log P_n \log \log P_n \log \log \log P_n} > \exists c > 0.$$

About the improvement of the constant  $c$ , see [9]. This completes the proof.  $\square$

Our method to show this type of results can be generalized by a kind of "linearity" in many cases. To explain this, we notice

**Theorem 4.** Let  $l$  be a fixed positive integer, and  $C_i$  ( $i = 1, 2, \dots, l$ ) be the real numbers with  $\sum C_i \neq 0$ . The sequence  $(\sum_{i=0}^{l-1} C_i \log P_{n+i})$  is not a.u.d. mod 1.

*Proof.* First, we consider the case  $(C \log P_n)$ . Without loss of generality, we may assume that  $C > 0$ . Then we write  $C \log P_n = \log_b P_n$  with a constant  $b > 1$ . To see the assertion, replace  $e$  with  $b$  in the proof of Theorem 2.

If  $l > 1$ , it suffice to note that

$$\sum_{i=0}^{l-1} C_i \log P_{n+i} - \log P_n \sum_{i=0}^{l-1} C_i = o(1).$$

This shows the assertion.  $\square$

**Theorem 5.** Let  $l$  be a fixed positive integer, and  $f_i$  ( $i = 1, 2, \dots, l$ ) be the positive real numbers. Then

$$\Delta^2 \log(P_n^{f_1} P_{n+1}^{f_2} \dots P_{n+l-1}^{f_l})$$

changes its sign infinitely many times.

*Proof.* Put

$$g(n) = n \left( \sum_{i=1}^l f_i \log P_{n+i-1} \right) - (n-1) \left( \sum_{i=1}^l f_i \log P_{n+i-2} \right)$$

$$f(n) = \sum_{i=1}^l f_i \log P_{n+i-1}.$$

By using Corollary 1 and Theorem 4, in a similar manner as in the proof of Theorem 3, we see the assertion. Here, we essentially used the positiveness of  $f_i$  ( $i = 1, 2, \dots, l$ ) in proving (C3).  $\square$

We expect that the conditions  $f_i > 0$  ( $i = 1, 2, \dots, l$ ) can be dropped.

Our method is applicable to a lot of arithmetic functions  $g(n)$ , that  $f(n) = 1/n \sum_{k \leq n} g(k)$  is not a.u.d. mod 1. For example, we can show similar assertions for the divisor function  $d(n) = \sum_{d|n} 1$  as

$$\frac{1}{n} \sum_{k=1}^n d(k) = \log n + (2\gamma - 1) + O\left(\frac{1}{\sqrt{n}}\right),$$

with the Euler constant  $\gamma$ . The proof for this case is easier, but the results do not seem well worthy of stating here.

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